Rings of Morita Contexts which are Maximal Orders

Pınar AYDOĞDU

Hacettepe University / TURKEY

(Joint work with E. Akalan, H. Marubayashi and B. Saraç)

NCRA IV
June 8-11, 2015

Background

A Morita context is a set $M = (R, V, W, S)$ and two maps θ and ψ, where

- V is an R-S bimodule,

Background

A Morita context is a set $M = (R, V, W, S)$ and two maps θ and ψ, where

- V is an R-S bimodule,
- W is an S-R bimodule.

Background

A Morita context is a set $M = (R, V, W, S)$ and two maps θ and ψ, where

- V is an R-S bimodule,
- W is an S-R bimodule.
- $\theta : V \otimes_S W \to R$ is an R-R bilinear map,

Background

A Morita context is a set $M = (R, V, W, S)$ and two maps θ and ψ, where

- V is an R-S bimodule,
- W is an S-R bimodule.
- $\theta : V \otimes_S W \to R$ is an R-R bilinear map,
- $\psi : W \otimes_R V \to S$ is an S-S bilinear map.

Background

Furthermore, the maps θ and ψ satisfy the associativity conditions that are required to make

$$T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$$

a ring.

Background

Furthermore, the maps θ and ψ satisfy the associativity conditions that are required to make

$$T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$$

a ring. T is called the ring of the Morita context.

Notation

For any $v \in V$ and $w \in W$,

Notation

For any $v \in V$ and $w \in W$,

- $\theta(v \otimes w)$ is denoted by vw,
- $\psi(w \otimes v)$ by wv.

Notation

For any $v \in V$ and $w \in W$,

- $\theta(v \otimes w)$ is denoted by vw,
- $\psi(w \otimes v)$ by wv.
- $Im(\theta)$ by VW,
- $Im(\psi)$ by WV.

Background

Let R be a prime Goldie ring and $Q(R)$ be its simple Artinian quotient ring.
Definition: Let I be an R-R bisubmodule of $Q(R)$. I is called fractional R-ideal if it satisfies

Background

Let R be a prime Goldie ring and $Q(R)$ be its simple Artinian quotient ring.
Definition: Let I be an R-R bisubmodule of $Q(R)$. I is called fractional R-ideal if it satisfies

1. I contains a regular element.

Background

Let R be a prime Goldie ring and $Q(R)$ be its simple Artinian quotient ring.
Definition: Let I be an R-R bisubmodule of $Q(R)$. I is called fractional R-ideal if it satisfies

1. I contains a regular element.
2. There exist regular elements $c_1, c_2 \in R$ such that $c_1 I \subseteq R$ and $Ic_2 \subseteq R$.

Background

A commutative ring
R is a Dedekind domain $\iff F(R) = \{I | I \text{ is a fractional } R\text{-ideal}\}$
is a group under multiplication.

Background

Let R be a right order in $Q(R)$.

Background

Let R be a right order in $Q(R)$. R is a maximal right order in $Q(R)$ $\Longleftrightarrow$ If there exists a right order S in $Q(R)$ and a regular element $c \in R$ such that either $cS \subseteq R$ or $Sc \subseteq R$ implies that $S = R$.

Background

Let I be a fractional R-ideal. Then
$O_l(I) = \{q \in Q(R) | qI \subseteq I\}$ is a left order of I.

Background

Let I be a fractional R-ideal. Then
$O_l(I) = \{q \in Q(R) | qI \subseteq I\}$ is a left order of I.
$O_r(I) = \{q \in Q(R) | Iq \subseteq I\}$ is a right order of I.

Background

Let I be a fractional R-ideal. Then
$O_l(I) = \{q \in Q(R) | qI \subseteq I\}$ is a left order of I.
$O_r(I) = \{q \in Q(R) | Iq \subseteq I\}$ is a right order of I.
Fact: R is a maximal order $\Longleftrightarrow O_l(I) = R = O_r(I)$ for every fractional R-ideal I.

Definition

Let I be a fractional R-ideal. $(R : I)_l = \{q \in Q(R) | qI \subseteq R\}$ and $(R : I)_r = \{q \in Q(R) | Iq \subseteq R\}$.

Definition

Let I be a fractional R-ideal. $(R : I)_l = \{q \in Q(R) | qI \subseteq R\}$ and $(R : I)_r = \{q \in Q(R) | Iq \subseteq R\}$.
$I_v = (R : (R : I)_l)_r$ and ${}_vI = (R : (R : I)_r)_l$.

Definition

Let I be a fractional R-ideal. $(R:I)_l = \{q \in Q(R) | qI \subseteq R\}$ and $(R:I)_r = \{q \in Q(R) | Iq \subseteq R\}$.
$I_v = (R:(R:I)_l)_r$ and ${}_vI = (R:(R:I)_r)_l$.
If $I_v = {}_vI$, then I is called a v-ideal or reflexive ideal.

If R is a maximal order, then $D(R) = \{v\text{-ideals}\}$ is a group under the multiplication $\circ$, where $I \circ J := (IJ)_v$.

Theorem[Marubayashi, Zhang and Yang, 1998]

$$T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$$

is a prime Goldie ring $\Longleftrightarrow$

Theorem[Marubayashi, Zhang and Yang, 1998]

$$T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$$

is a prime Goldie ring $\Longleftrightarrow$

1. R and S are prime Goldie rings,

Theorem[Marubayashi, Zhang and Yang, 1998]

$$T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$$

is a prime Goldie ring $\Longleftrightarrow$

1. R and S are prime Goldie rings,
2. $vW = 0 \Rightarrow v = 0$ and

Theorem[Marubayashi, Zhang and Yang, 1998]

$$T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$$

is a prime Goldie ring $\Longleftrightarrow$

1. R and S are prime Goldie rings,
2. $vW = 0 \Rightarrow v = 0$ and $Vw = 0 \Rightarrow w = 0$,

Theorem[Marubayashi, Zhang and Yang, 1998]

$$T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$$

is a prime Goldie ring $\Longleftrightarrow$

1. R and S are prime Goldie rings,
2. $vW = 0 \Rightarrow v = 0$ and $Vw = 0 \Rightarrow w = 0$,
3. $VsW = 0 \Rightarrow s = 0$.

From now on we assume that

$$T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$$

is a prime Goldie ring.

From now on we assume that

$$T = \begin{pmatrix} R & V \\ W & S \end{pmatrix}$$

is a prime Goldie ring.
Then

$$Q(T) = \begin{pmatrix} Q(R) & Q(V) \\ Q(W) & Q(S) \end{pmatrix}$$

is the quotient ring of T, where
$Q(V) = VQ(S) = Q(R)V$ and $Q(W) = WQ(R) = Q(S)W$.

Definition

Let V_1 be an R-S submodule of $Q(V)$. V_1 is called a fractional R-S module

$\Longleftrightarrow$

1. $V_1 Q(S) = Q(V) = Q(R) V_1$
2. There exist regular elements $c \in R$ and $d \in S$ such that $cV_1 \subseteq V$ and $V_1 d \subseteq V$.

Definition

Let V_1 be a fractional R-S-module. Then
$O_l(V_1) = \{q \in Q(R) | qV_1 \subseteq V_1\}$ and
$O_r(V_1) = \{q \in Q(S) | V_1 q \subseteq V_1\}$

Definition

Let V_1 be a fractional R-S-module. Then
$O_l(V_1) = \{q \in Q(R) | qV_1 \subseteq V_1\}$ and
$O_r(V_1) = \{q \in Q(S) | V_1 q \subseteq V_1\}$
${}_R V_S$ is a maximal module in $Q(V)$ if $O_l(V_1) = R$ and $O_r(V_1) = S$ for every fractional R-S-module V_1 of $Q(V)$.

Definition

Let V_1 be a fractional R-S submodule of $Q(V)$.
$(S : V_1)_l = \{w' \in Q(W) | w' V_1 \subseteq S\}$.

Definition

Let V_1 be a fractional R-S submodule of $Q(V)$.
$(S : V_1)_l = \{w' \in Q(W) | w'V_1 \subseteq S\}$.
$V_{1v} := (S : (S : V_1)_l)_r$ and ${}_vV_1 := (R : (R : V_1)_r)_l$

Definition

Let V_1 be a fractional R-S submodule of $Q(V)$.
$(S : V_1)_l = \{w' \in Q(W) | w'V_1 \subseteq S\}$.
$V_{1v} := (S : (S : V_1)_l)_r$ and ${}_vV_1 := (R : (R : V_1)_r)_l$
If $V_{1v} = {}_vV_1$, then V_1 is called a v-(R, S)-module.

Theorem[Marubayashi, Zhang and Yang, 1998]

TFAE:

1. T is a maximal order in $Q(T)$.
2. (i) R and S are maximal orders in $Q(R)$ and $Q(S)$, respectively;
   (ii) $(R : W)_l = V = (S : W)_r$ and $(R : V)_r = W = (S : V)_l$.

Theorem

TFAE:

1. T is a maximal order in $Q(T)$.
2. (i) V is an R-S maximal module in $Q(V)$ and W is an S-R maximal module in $Q(W)$;
   (ii) $(R : W)_l = V = (S : W)_r$ and $(R : V)_r = W = (S : V)_l$.
3. (i) V is an R-S maximal module in $Q(V)$ and W is an S-R maximal module in $Q(W)$;
   (ii)$(VW)_v = R =_v (VW)$ and $(WV)_v = S =_v (WV)$;
   (iii) $V_v = V = {}_vV$ and $W_v = W = {}_vW$.

Theorem

Suppose that T is a maximal order in $Q(T)$. Then there exists a 1-1 correspondence between $D(V)$ and $D(R)$ given by:

$$V_1 \to (V_1 W)_v \text{ and } I \to (IV)_v$$

Theorem

Suppose that T is a maximal order in $Q(T)$. Then there exists a group isomorphism between $D(R)$ and $D(T)$ given by

$$I \longleftrightarrow \begin{pmatrix} I & (IV)_v \\ (WI)_v & (WIV)_v \end{pmatrix}$$

Applications

Asano order: A prime Goldie ring in which each non-zero ideal is invertible.

Applications

Asano order: A prime Goldie ring in which each non-zero ideal is invertible.

Lemma: Suppose that R and S are Asano orders in $Q(R)$ and $Q(S)$, respectively.

Applications

Asano order: A prime Goldie ring in which each non-zero ideal is invertible.

Lemma: Suppose that R and S are Asano orders in $Q(R)$ and $Q(S)$, respectively. Then

1. For each fractional (R, S)–module V' in $Q(V)$ we have $(V')^{-1}V' = S$ and $V'(V')^{-1} = R$.

Applications

Asano order: A prime Goldie ring in which each non-zero ideal is invertible.

Lemma: Suppose that R and S are Asano orders in $Q(R)$ and $Q(S)$, respectively. Then

1. For each fractional (R, S)–module V' in $Q(V)$ we have $(V')^{-1}V' = S$ and $V'(V')^{-1} = R$.
2. For each fractional (S, R)–module in $Q(W)$ we have $(W')^{-1}W' = R$ and $W'(W')^{-1} = S$.

Definition

V is an (R, S)–Asano module in $Q(V)$ if for each integral (R, S)–module V', $(V')^{-1}V' = S$ and $V'(V')^{-1} = R$.

Definition

V is an (R, S)–Asano module in $Q(V)$ if for each integral (R, S)–module V', $(V')^{-1}V' = S$ and $V'(V')^{-1} = R$.
Similarly we can define an (S, R)–Asano module W in $Q(W)$.

Definition

V is an (R, S)–Asano module in $Q(V)$ if for each integral (R, S)–module V', $(V')^{-1}V' = S$ and $V'(V')^{-1} = R$. Similarly we can define an (S, R)–Asano module W in $Q(W)$.

- It follows from Lemma that if R and S are Asano orders in $Q(R)$ and $Q(S)$, respectively, then V is an (R, S)–Asano module in $Q(V)$ and W is an (S, R)–Asano module in $Q(W)$.
- Moreover, it is easy to see that V is an (R, S)–Asano module in $Q(V)$, then it is an (R, S)–maximal module in $Q(V)$.
- An analogous result can be given for W.

Suppose that $VW = R$ and $WV = S$.

Suppose that $VW = R$ and $WV = S$.
Then there is a one-to-one correspondence between the set of all fractional R–ideals and the set of all fractional (R, S)–modules in $Q(V)$,

Suppose that $VW = R$ and $WV = S$.
Then there is a one-to-one correspondence between
the set of all fractional R–ideals and the set of all fractional (R, S)–modules in $Q(V)$,
which is given by :

Suppose that $VW = R$ and $WV = S$.
Then there is a one-to-one correspondence between
the set of all fractional R–ideals and the set of all fractional (R, S)–modules in $Q(V)$,
which is given by :
$I \longrightarrow IV = V'$ and $V' \longrightarrow V'W$,
where I is a fractional R–ideal and V' is a fractional (R, S)–module in $Q(V)$.

Theorem

The following conditions are equivalent:

1. T is an Asano order in $Q(T)$.

Theorem

The following conditions are equivalent:

1. T is an Asano order in $Q(T)$.
2. (i) R and S are Asano orders in $Q(R)$ and $Q(S)$, respectively, and
   (ii) $VW = R$ and $WV = S$.

Theorem

The following conditions are equivalent:

1. T is an Asano order in $Q(T)$.
2. (i) R and S are Asano orders in $Q(R)$ and $Q(S)$, respectively, and
   (ii) $VW = R$ and $WV = S$.
3. (i) V is an (R, S)–Asano module in $Q(V)$ and W is an (S, R)–Asano module in $Q(W)$, and
   (ii) $VW = R$ and $WV = S$.

Dedekind order: A prime Goldie ring which is a maximal order and hereditary.

Dedekind order: A prime Goldie ring which is a maximal order and hereditary.

Definition: V is called an (R,S)–Dedekind module in $Q(V)$ if

- V is an (R,S)–maximal module in $Q(V)$, and
- every left R–submodule of V is projective and every right S–submodule of V is projective.

Dedekind order: A prime Goldie ring which is a maximal order and hereditary.
Definition: V is called an (R, S)–Dedekind module in $Q(V)$ if

- V is an (R, S)–maximal module in $Q(V)$, and
- every left R–submodule of V is projective and every right S–submodule of V is projective.

Similarly we can define an (S, R)–Dedekind module W in $Q(W)$.

Theorem

The following three conditions are equivalent:

1. T is a Dedekind order in $Q(T)$.

Theorem

The following three conditions are equivalent:

1. T is a Dedekind order in $Q(T)$.
2. (i) R is a Dedekind order in $Q(R)$ and S is a Dedekind order in $Q(S)$, respectively, and
   (ii) $VW = R$ and $WV = S$.

Theorem

The following three conditions are equivalent:

1. T is a Dedekind order in $Q(T)$.
2. (i) R is a Dedekind order in $Q(R)$ and S is a Dedekind order in $Q(S)$, respectively, and
   (ii) $VW = R$ and $WV = S$.
3. (i) V is an (R, S)–Dedekind module in $Q(V)$ and W is an (S, R)–Dedekind module in $Q(W)$, and
   (ii) $VW = R$ and $WV = S$.

Definition[Akalan, 2008]

A prime Goldie ring R is called a Generalized Dedekind prime (G-Dedekind, for short) ring if

- R is a maximal order and
- Every v-ideal is invertible.

Conjecture

T is a G-Dedekind prime ring $\Longleftrightarrow$

1. R and S are G-Dedekind prime rings,
2. $(R : W)_l = V = (S : W)_r$ and $(R : V)_r = W = (S : V)_l$.

Acknowledgement

This work has been supported by TÜBITAK (Project no: 113F032). We would like to thank TÜBITAK for their support.